Birzeit University-Mathematics Department
Calculus II-Math 132

First Exam Second Semester 2015/2016
Name(Arabic):......Solution.. Number:..................
Instructor of Discussion(Arabic):................................ Section:...................

Question 1.(38 points) Circle the correct answer:

(1) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}} dx$

- (a) Diverges.
- (b) Converges to 0
- (c) Converges to 1
- (d) $\boxed{\text{Converges to 2}}$

(2) $\int_1^{\infty} \frac{\ln x}{x^3} dx$

- (a) $\boxed{\text{Converges by direct comparison with } \int_1^{\infty} \frac{dx}{x^2}}$
- (b) Diverges by direct comparison with $\int_1^{\infty} \frac{dx}{x}$
- (c) Diverges by direct comparison with $\int_1^{\infty} \ln x dx$
- (d) Converges by direct comparison with $\int_1^{\infty} \frac{dx}{x^3}$

(3) The sequence $a_n = \frac{\ln(2n)}{\ln(3n)}$

- (a) $\boxed{\text{Converges to 1}}$
- (b) Diverges.
- (c) Converges to $\frac{\ln 2}{\ln 3}$
- (d) Converges to $\frac{2}{3}$

(4) $\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$

- (a) Converges by limit comparison with $\int_1^{\infty} \frac{dx}{x}$
- (b) Diverges by direct comparison with $\int_1^{\infty} \frac{dx}{\sqrt{x^2+1}}$
- (c) $\boxed{\text{Diverges by direct comparison with } \int_1^{\infty} \frac{dx}{x}}$
- (d) Converges by limit comparison with $\int_1^{\infty} \frac{dx}{x^2}$

(5) The series $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$

(a) $\boxed{\text{Diverges by nth term test.}}$

(b) Converges by ratio test.

(c) Diverges by ratio test.

(d) Converges by integral test.

(6) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

(a) Converges by nth term test.

(b) Diverges by nth term test.

(c) $\boxed{\text{Converges by root test.}}$

(d) Diverges by root test.

(7) The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-1}$

(a) $\boxed{\text{Converges conditionally.}}$

(b) Converges absolutely.

(c) Diverges.

(d) None of the above.

(8) $\sum_{n=1}^{\infty} \frac{1}{n-\ln n}$

(a) Converges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

(b) $\boxed{\text{Diverges by limit comparison test with } \sum_{n=1}^{\infty} \frac{1}{n}}$

(c) Converges by ratio test.

(d) Diverges by ratio test.

(9) $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots =$

(a) 1

(b) $\boxed{-1}$

(c) 0

(d) π

(10) The error in the approximation $\cos x \approx 1 - \frac{x^2}{2!}$, $|x| < 0.1$ using the alternating series estimation theorem will be less than

(a) $\frac{1}{4!}$

(b) $\frac{(10)^4}{4!}$

(c) $\boxed{\frac{10^{-4}}{4!}}$

(d) 10^{-4}

(11) The sequence $a_n = (2^n + 3^n)^{1/n}$

(a) Converges to 0

(b) Converges to 2

(c) Diverges.

(d) $\boxed{\text{Converges to 3.}}$

(12) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{2}{3^n}\right) =$

(a) $\boxed{2}$

(b) $\frac{3}{2}$

(c) 3

(d) 5

(13) The nth partial sum of the series $\sum\limits_{n=1}^{\infty} \frac{4}{(2n-1)(2n+1)}$ is

(a) $s_n = \frac{4}{(2n-1)(2n+1)}$

(b) $s_n = 1 - \frac{2}{2n+1}$

(c) $\boxed{s_n = 2 - \frac{2}{2n+1}}$

(d) $s_n = \frac{2}{2n+1}$

(14) The series $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

(a) $\boxed{\text{Converges by ratio test.}}$

(b) Diverges by ratio test.

(c) Converges by integral test.

(d) Diverges by nth term test.

(15) The series $\sum_{n=1}^{\infty} (\ln x)^n$

(a) Converges for all x

(b) Converges for $-1 < x < 1$

(c) Converges for $-e < x < e$

(d) $\boxed{\text{Converges for } e^{-1} < x < e}$

(16) The Maclaurin series generated by the function $f(x) = 2^x$ is

(a) $\sum_{n=0}^{\infty} \frac{\ln 2}{n!} x^n$

(b) $\boxed{\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n}$

(c) $\sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!} x^n$

(d) $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

(17) The binomial series generated by the function $f(x) = (1+x)^{-1/3}$ is

(a) $1 - \frac{1}{3}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots$

(b) $1 - \frac{1}{3}x + \frac{1}{6}x^2 - \frac{3}{8}x^3 + \cdots$

(c) $\boxed{1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \cdots}$

(d) $1 - x + x^2 - x^3 + \cdots$

(18) The series $\sum_{n=1}^{\infty} \frac{n^{1/n}}{n}$

(a) Diverges by nth term test.

(b) Converges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) $\boxed{\text{Diverges by limit comparison test with } \sum_{n=1}^{\infty} \frac{1}{n}}$

(d) Diverges by limit comparison test with $\sum_{n=1}^{\infty} n^{1/n}$

(19) The Maclaurin series generated by $\frac{x^2}{(1-x)^2}$ is

(a) $\boxed{\sum_{n=1}^{\infty} n x^{n+1}}$

(b) $\sum_{n=1}^{\infty} n x^{n+2}$

(c) $\sum_{n=1}^{\infty} (n+1) x^{n+2}$

(d) $\sum_{n=0}^{\infty} (n+1)(n+2) x^{n+2}$

Question 2(12 points) Find the interval and radius of convergence of the series

$$\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$$

Give full details.

Applying the ratio test, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{n+1}}{(n+1)\ln^2(n+1)} \frac{n\ln^2 n}{|x|^n} = |x| \frac{n}{n+1} \frac{\ln^2 n}{\ln^2(n+1)} \to |x| \quad \textbf{(2 points)}$$

The series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. (1 point)

If $x = 1$, then we get the series

$$\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$$

We apply the integral test

$$\int_2^{\infty} \frac{dx}{x\ln^2 x} = \lim_{a\to\infty} \int_2^a \frac{dx}{x\ln^2 x} = \lim_{a\to\infty} \left.\frac{-1}{\ln x}\right|_2^a = \lim_{a\to\infty} \left(-\frac{1}{\ln a} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2} \quad \textbf{(2 points)}$$

Therefore, the series converges at $x = 1$ by integral test.(1 point)

If $x = -1$, we have the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln^2 n}$$

We apply the alternating series test:

- $u_n = \frac{1}{n\ln^2 n} > 0$ **(1 point)**
- $u_{n+1} = \frac{1}{(n+1)\ln^2(n+1)} < u_n = \frac{1}{n\ln^2 n}$ **(1 point)**
- $u_n = \frac{1}{n\ln^2 n} \to 0$ **as** $n \to \infty$ **(1 point)**

So, the series converges by alternating series test at $x = -1$ (1 point)

Interval of convergence $-1 \le x \le 1$ (1 point)

Radius of convergence $R = 1$ (1 point)

Question 3(10 points) Given that $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$, $-1 < t < 1$.

(a) Find the Maclaurin series of the function $\frac{1}{1+t^2}$.

$$\mathbf{\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - t^{10} \cdots} \quad \textbf{(2 points)}$$

(b) Use (a) and integration to find the Maclaurin series of $\tan^{-1} x$.

$$\int_0^{\mathbf{x}} \frac{\mathbf{1}}{\mathbf{1+t^2}} = \int_0^{\mathbf{x}} (\mathbf{1 - t^2 + t^4 - t^6 + t^8 - t^{10}} + \cdots)\,\mathbf{dx}$$

$$\tan^{-1} \mathbf{t}\Big|_0^{\mathbf{x}} = \left(\mathbf{t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \frac{x^{11}}{11}} + \cdots \right|_0^{\mathbf{x}}$$

$$\tan^{-1} \mathbf{x} = \mathbf{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}} - \cdots \qquad \textbf{(3 points)}$$

(c) Use (b) to approximate $\int_0^1 \frac{\tan^{-1} x}{x} dx$ with an error of magnitude less than 0.01.

$$\frac{\tan^{-1} x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \frac{x^8}{9} - \frac{x^{10}}{11} + \cdots \quad \textbf{(2 points)}$$

$$\begin{aligned}
\int_0^1 \frac{\tan^{-1} x}{x} dx &= \int_0^1 \left(1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \frac{x^8}{9} - \frac{x^{10}}{11} + \cdots \right) dx \\
&= x - \frac{x^3}{9} + \frac{x^5}{25} - \frac{x^7}{49} + \frac{x^9}{81} - \frac{x^{11}}{121} + \cdots \Big|_0^1 \\
&= 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \frac{1}{121} + \cdots \quad \textbf{(2 points)}
\end{aligned}$$

Therefore,

$$\int_0^1 \frac{\tan^{-1} x}{x} dx \approx 1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} \quad \textbf{(1 point)}$$

Using alternating series estimation theorem,

$$|\text{error}| < \frac{1}{100}$$